A Landau–Ginzburg Language for Spectral Sequences

1 The Story in Broad Strokes

By "broad strokes," I mean: I will tell the story first without specifying a specific geometric context—that is, without saying what stacks mean or what commutativity means. A bird's-eye view of the construction.

1.1 A Cast of Geometric Objects

- **The Multiplicative Group Scheme**: A commutative group stack $\mathbb{G}_m$.
- **The Cartier Stack**: A pointed stack over $B\mathbb{G}_m$ with
  - underlying $\mathbb{G}_m$-equivariant stack $\mathbb{A}/\mathbb{G}_m \to B\mathbb{G}_m$,
  - basepoint $0 : B\mathbb{G}_m \to \mathbb{A}$.
- **A Quantized Stack**: A stack X equipped with a map $f : X \to \mathbb{A}/\mathbb{G}_m$.

There are two physical perspectives on how to interpret this data.

1.2 Physical Interpretation of a Quantized Stack

1. $\mathbb{A} = \mathrm{Spec}(\daleth[\hbar])/(\text{scaling in } \hbar)$. Then f exhibits X as an equivariant deformation quantization of the central fiber

$$X_0 := X \times_{\mathbb{A}/\mathbb{G}_m} B\mathbb{G}_m.$$

That it is $\hbar$-scaling equivariant means we are constructing a theory where we mostly care about what happens at $\hbar = 1$ and $\hbar = 0$. In other words, this is a theory that is allowed to forget detailed information about what happens when we go from $\hbar = 0.1$ to $\hbar = 0.15$.

2. Alternatively, f can be thought of as a "potential" $V = d\mathcal{S}$: the derivative of an action functional on a phase space of states X. The central fiber X_0 again has a physical interpretation: it is the "classical locus"—the space of states carved out by the Euler–Lagrange equations.

1.3 The Central Fiber Diagram

Staring at the diagram defining the central fiber:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ B\mathbb{G}_m \arrow[r, "0"'] & \mathbb{A}/\mathbb{G}_m \end{tikzcd}$$

We then perform a formal completion of the entire picture:

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \widehat{X}_{X_0} \arrow[r, hook] \arrow[d, "\hat{f}"] & X \arrow[d, "f"] \\ B\mathbb{G}_m \arrow[r, "0"'] & \widehat{\mathbb{A}/\mathbb{G}_m} \arrow[r, hook] & \mathbb{A}/\mathbb{G}_m \end{tikzcd}$$

By switching from the intersection $f \cap 0$ to the formally completed intersection $\hat{f} \cap \hat{0}$, we pass from the world of "all geometry" to the world of "formal geometry" (all of this over $B\mathbb{G}_m$).

Once in the world of formal geometry, we can use the formal moduli problem form of Koszul duality to turn questions about geometry on $\widehat{X}$ into questions about algebra on X_0.

2 Matrix Factorizations

We now give some ways to describe the category $MF(X, f)$ (or more precisely, $MF(\widehat{X}, \hat{f})$) of matrix factorizations on X for the potential f.

2.1 Quantized Complete Sheaves

- $\widehat{X}$ gives rise to the category $\mathrm{Coh}(\widehat{X})$.
- $\hat{f}$ equips it with an action by the category $\mathrm{Coh}(\widehat{\mathbb{A}/\mathbb{G}_m})$.

Here, Coh denotes some notion of formally complete sheaves on formal moduli stacks (details omitted in this introduction).

That is, **matrix factorizations** are the data of this complete **filtered** category. Fixing the total space $\widehat{X}$, changing the potential f changes the filtration.

2.2 Representations of Automorphisms

A characteristic property of formal moduli problems under X_0 is that we have equivalences

$$\widehat{X} \simeq \beta_{X_0} \,\mathfrak{aut}(X_0/X),$$

where $\mathfrak{aut}(X_0/-)$ is the formal group of symmetries of the base point of $\widehat{X}$, and β is an inverse to that functor.

Thus,

$$\mathrm{Coh}(\widehat{X}) \simeq \mathrm{Rep}(\mathfrak{aut}(X_0/X); \mathrm{Coh}(X_0)).$$

If we remember the map $X_0 \to B\mathbb{G}_m$, then this category acquires a grading.

The main construction here actually uses Koszul duality on the downstairs arrow, which tells us there is a categorical action by the formal group

$$\mathfrak{aut}(B\mathbb{G}_m/(\mathbb{A}/\mathbb{G}_m))$$

on the category $\mathrm{Coh}(X_0)$.

This categorical representation

$$\mathrm{Coh}(X_0) \in \mathrm{Rep}(\mathfrak{aut}(0/\text{Cartier Stack}); 2\mathrm{QCoh}(B\mathbb{G}_m))$$

is also said to be the matrix factorization category of $(\widehat{X}, \hat{f})$.

2.3 Equivalence of Perspectives

That these two different 2-categorical objects are both "matrix factorization categories" makes sense because there is an equivalence of theories:

$$\mathrm{Rep}(\mathfrak{aut}(0/\text{Cartier Stack}); 2\mathrm{QCoh}(B\mathbb{G}_m)) \simeq 2\mathrm{Coh}(\widehat{\mathbb{A}/\mathbb{G}_m}).$$

3 Next Orders of Business

- Example/Application: the Koszul duality diagram in Hodge theory.
- Koszul-dual description of synthetic spectra.
- The physical interpretation.