Coherent Obstruction Theory of Spectral Algebras

1 Square-Zero extensions of discrete rings Linearize

A central phenomenon uncovered in the deformation/obstruction theory of discrete rings (in the tradition of Artin, Illusie, Hartshorne, ...) is that the a priori *geometric/non-linear* theory of square-zero extensions *linearizes*:

$$\{\text{Square zero extensions of } R \text{ by } I\} \simeq \mathrm{Map}_{D(R)}(\mathbb{L}_R, \Sigma I).$$

To drive home the point about linearity, observe that the LHS is a priori a groupoid. The RHS, being mapping space in a stable ∞-category, has the structure of an E_∞ group. That is, this equivalence tells us that it makes sense to "add" square zero extensions and "scale" square zero extensions by elements of R — that there's an operation building new square-zero extensions from old ones by taking R-linear superpositions/weighted sums.

2 "Bounded" extensions of ring spectra also linearize

The same phenomenon occurs when the square-zero ideal I has bounded Postnikov amplitude, by work of Basterra-Mandell and Lurie.

Theorem 2.1 (Lurie, (HA Thm. 7.4.1.23, Defn. 7.4.1.18))**.**

$$\{\text{Square zero extensions } A \to R \text{ by } I\} \simeq \mathrm{Map}_{D(R)}(\mathbb{L}_R, \Sigma I),$$

granted that A is connective and I has homotopy groups concentrated in degree range $[n, 2n]$.

This note explores what happens when the strong connectivity and truncation constraints here do not hold. Examples: MU, $THH(\mathbb{F}_p)$, $\cdots$

3 Coherent square-zero extensions

Definition 3.1 (Square Zero Extensions)**.** Let

- $\mathcal{O}$ be a coherent stable operad (in the sense of HA chapter 5),
- $V^{\otimes} \in CAlg(Mod_{Sp}(Pr^L))$ be a presentably symmetric monoidal stable ∞-category,
- $A \in Alg^{\mathcal{O}}(V)$ be an $\mathcal{O}$-algebra in V.

A (coherent) *square-zero extension* of A is the data of a pair $(\mathcal{R}, \theta)$, where

- $\mathcal{R} \in Alg^{\mathcal{O}}(\widehat{Fil_V})$, i.e. an $\mathcal{O}$-algebra in complete filtered objects of V,
- $\theta : gr^0(\mathcal{R}) \xrightarrow{\sim} A$ an equivalence in $Alg^{\mathcal{O}}(V)$,
- and $\mathcal{R}$ is required to satisfy the additional property $gr^q(\mathcal{R}) = 0$ for all $q \notin \{0, 1\}$.

In this situation, we say that $(\mathcal{R}, \theta)$ is a square-zero extension of A by $gr^1\mathcal{R}$.

Footnotes

- **Presentability:** I'm not sure if the presentability assumption on V is strictly necessary. It's possible that I'm asking for it because (1) the simple expression $Fil_V \simeq Fil \otimes_{Sp} V$ simplifies our arguments, and (2) all our examples will be presentable.
- **Attribution:** I learned this from Tyler Lawson, but it might have a longer history, e.g. in model-structures approaches to Smith Ideals.

3.1 What do Square Zero Extensions Look Like?

Remark 3.2 (Underlying filtrations of square zero extensions)**.** The combination of the conditions

- that the associated graded of $\mathcal{R}$ is concentrated in degrees 0 and 1,
- that the filtration on $\mathcal{R}$ is complete

forces the underlying filtered object of $\mathcal{R}$ to be of the form

$$\cdots \xrightarrow{\sim} 0 \to 0 \to I \xrightarrow{f} R \xrightarrow{\sim} R \xrightarrow{\sim} \cdots$$

where I sits in degree 1, mapping to an R in degree 0 via the map f, and we have an equivalence

$$\mathrm{coFib}(f) \simeq A.$$

The objects I and R here have additional structures:

- $gr^1(\mathcal{R}) \simeq I$ has the structure of a module over the algebra $gr^0(\mathcal{R}) \simeq A$, hence, the object $I \in V$ sitting in degree 1 has the structure of an A-module. Here and already before this point, we've used extensively that gr^0 and $gr^\bullet$ are symmetric monoidal functors, among other salient properties about filtrations and gradings. These salient properties can be summarized by that $gr : Fil \to Gr$ is the universal perverse schober, a theorem of Gammage-Hilburn-Mazel-Gee.

- the $\mathcal{O}$-algebra structure on $\mathcal{R}$ induces a $\mathcal{O}$-algebra structure on the 0-th degree term of the filtration $\mathcal{R}(0) \simeq R$, as the functor $(-)(0)$ is lax symmetric monoidal. We can also see this from that the "converges-to functor" $(-)(-\infty)$ is lax symmetric monoidal.

In short, a square-zero extension of an $\mathcal{O}$-algebra A by an ($\mathcal{O}$-operadic) A-module I is given by

- an $\mathcal{O}$-algebra R,
- a map of algebras $R \to A$ whose kernel is the A-module I,
- some additional coherence data. We'll see later that this data precisely encodes what it means for I to be "square-zero" in higher algebra.

Remark 3.3 (Square zero extensions as 1-term coherent cochains)**.** By the correspondence between complete filtered spectra and homotopy coherent cochains (whose strongest modern version was proved by Ariotta), the above description of the filtration tells us that a coherent extension of A is an algebra in coherent cochains whose underlying cochain object looks like

$$\cdots \xrightarrow{\simeq} 0 \xrightarrow{\simeq} 0 \xrightarrow{\simeq} A \xrightarrow{d} \Sigma I \to 0 \xrightarrow{\simeq} 0 \xrightarrow{\simeq} \cdots$$

This encodes the idea of a *derivation* valued in I.
TODO: explain the degree shift in terms of the universal property of $\mathbb{S}(1)[1] \in Fil$.

4 Linearization of coherent square-zero extensions

For discrete rings, we saw that

- the **non-linear/geometric/"commutative"** data of a square-zero thickening ring map
- is converted to
- the **linear/representation-theoretic/"non-commutative"** data.

By our discussion above, we see that

- the **non-linear** data is the filtered algebra description (encodes coherences on a ring map),
- the **linear data** is the coherent cochains description (encodes structure on a derivation),
- the conversion is performed by the **Ariotta correspondence**.

The **lesson** we learn from this is that by passing to a more coherent version of the notion of square-zero extension we get to classify square-zero extensions of arbitrary ring spectra, using an argument where neither connectivity or truncatedness assumptions are needed. Contrast this to the situation in HA where square-zeroness is defined for maps with connective fibers, and there it's defined as a condition $\pi_0(I)$.